1 Concave and convex functions

Definition 1 *A function f defined on the convex set $C \subset \mathbb{R}^n$ is called* ***concave*** *if for every $x_1, x_2 \in C$ and $0 \leq t \leq 1$, we have*

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Definition 2 *A function f defined on the convex set $C \subset \mathbb{R}^n$ is called* ***strictly concave*** *if for every $x_1 \neq x_2$,and $0 < t < 1$, we have*

$$f(tx_1 + (1-t)x_2) > tf(x_1) + (1-t)f(x_2).$$

Remark 3 *If f is (strictly) concave then $g \equiv -f$ is a (strictly) convex function. We will henceforth concentrate on concave functions. All the results will also obtain with the obvious modifications for convex functions.*

2 Concave functions of one variable

Theorem 4 ***(Continuity of concave functions)***
Let f be concave function on the convex set $C \subset \mathbb{R}$. Then f is continuous on the interior of C.

Theorem 5 *Let f be a differentiable function on the open convex set $C \subset \mathbb{R}$. It is concave if and only if for every $x_0, x \in C$, we have*

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0).$$

It is strictly concave if and only if the inequality is strict for $x \neq x_0$.

Theorem 6 *Let f be a differentiable function on the open convex set $C \subset \mathbb{R}$. It is concave (strictly concave) if and only if f' is a nonincreasing (decreasing) function.*

Theorem 7 ***(Concavity for C^2 functions)***
Let f be a function on the open convex set $C \subset \mathbb{R}$. Suppose that f" exists on C. Then f is a concave function if and only if f"$(x) \leq 0$ for every $x \in C$. If f"$(x) < 0$ for every $x \in C$, then f is strictly concave.

3 Concave function of several variables

Definition 8 *For any function f on C and any $\alpha \in \mathbb{R}$ the set $U(f, \alpha)$ defined by*

$$U(f, \alpha) = \{x : x \in C, f(x) \geq \alpha\}$$

*is called the **upper level** (or contour set of. The set*

$$L(f, \alpha) = \{x : x \in C, f(x) \leq \alpha\}$$

*is called the **lower level** (or contour) set of f. The set*

$$Y(f, \alpha) = \{x : x \in C, f(x) = \alpha\}$$

*is called the **level surface** of f at α.*

Corollary 9 *Let f be a concave (convex) function on $C \subset \mathbb{R}^n$. Then its upper (lower) level sets are convex sets for every real number α.*

Theorem 10 (***Concavity for*** C^1***functions***)
Let f be a differentiable function on the open convex set $C \subset \mathbb{R}^n$. It is concave if and only if for every $x_0, x \in C$ we have

$$f(x) - f(x_0) \leq \nabla f(x_0) \cdot (x - x_0) .$$

It is strictly concave if and only if the inequality is strict for $x \neq x_0$.

In order to state the condition for concavity of a function in second order partial derivatives, recall the following definitions and results.
For convenience and later reference, we represent here the matrix notation used further on as well as several characterizations of quadratic forms.

Suppose A is an $n \times n$ symmetric matrix of the following form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \dots & a_{3n} \\ \dots & & & & \dots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{pmatrix}$$

Definition 11 *A* ***quadratic form*** *is a function $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$Q_A(y) = y \cdot Ay = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} y_i y_j$$

where A is an $n \times n$ symmetric matrix and $y \in \mathbb{R}^n$.

Definition 12 *Suppose that A is an $n \times n$ symmetric matrix and that $Q_A(y) = y \cdot Ay$ is the quadratic form associated with A. Then A and Q_A are called:*

1. **positive semidefinite** if $Q_A(y) = y \cdot Ay \geq 0$ for all $y \in \mathbb{R}^n$;
2. **positive definite** if $Q_A(y) = y \cdot Ay > 0$ for all $y \in \mathbb{R}^n, y \neq 0$;
3. **negative semidefinite** if $Q_A(y) = y \cdot Ay \leq 0$ for all $y \in \mathbb{R}^n$;
4. **negative definite** if $Q_A(y) = y \cdot Ay < 0$ for all $y \in \mathbb{R}^n, y \neq 0$;
5. **indefinite** if $Q_A(y) = y \cdot Ay < 0$ for some $y \in \mathbb{R}^n$ and $Q_A(y) > 0$ for other $y \in \mathbb{R}^n$.

Suppose A is a $n \times n$ symmetric matrix. Define Δ_k to be the determinant of the upper left-hand corner $k \times k$ submatrix of A for $1 \leq k \leq n$. The determinant Δ_k is called the k-th **leading principal minor** of A with $\Delta_1 = a_{11}$

$$\Delta_2 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

and finally $\Delta_n = \det A$.

Theorem 13 *If A is an $n \times n$ symmetric matrix and if Δ_k is the k-th* leading principal minor *of A for $1 \leq k \leq n$, then*

1. *A is **positive definite** if and only if $\Delta_k > 0$ for $k = 1, 2, ..., n$;*
2. *A is **negative definite** if and only if $(-1)^k \Delta_k > 0$ for $k = 1, 2, ..., n$ (that is, the* leading principal minors *alternate in sign with $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0$, etc.).*

To test semidefiniteness we have to evaluate all *principal minors*, and therefore we have to examine a greater number of determinants. If A is a quadratic matrix of order n and we wipe out (arbitrary) r of the rows and the **corresponding** r columns as well, the resulting $(n-r) \times (n-r)$ submatrix is called a **principal minor** $\tilde{\Delta}_k$ is the k-th principal minor of A with $k = n-r$. We have the following theorem:

Theorem 14 *If A is an $n \times n$ symmetric matrix and if $\tilde{\Delta}_k$ is the k-th principal minor of A for $1 \leq k \leq n$, then*

1. *A is **positive semidefinite** if and only if $\tilde{\Delta}_k \geq 0$ for all principal minors of dimension k and $k = 1, 2, ..., n$;*

2. *A is **negative semidefinite** if and only if* $(-1)^k\tilde{\Delta}_k \geq 0$ *for all principal minors of dimension* k *and for* $k = 1, 2, ..., n$ *(that is, the principal minors alternate in sign with* $\tilde{\Delta}_1 \leq 0, \tilde{\Delta}_2 \geq 0, \tilde{\Delta}_3 \leq 0,$ *etc.).*

3. *A positive (negative) semidefinite matrix* A *is positive (negative) defnite if and only if* A *is a nonsingular matrix.*

With this in mind, we have:

Theorem 15 ***(Concavity for*** C^2 ***functions)***
Let f *be a twice differentiable function on an open convex set* $C \subset \mathbb{R}^n$*. Then* f *is concave if and only if its Hessian matrix of the second partial derivatives is negative semidefinite for every* $x \in C$*. That is, for every* $x \in C$ *and* $y \in \mathbb{R}^n$*, we have*

$$y^T H(x) y \leq 0$$

If $H(x)$ *is negative definite for every* $x \in C$*, then* f *is strictly concave.*

Now we state a few results on the closedness of concave functions under functional operations.

Theorem 16 *Let* f *be concave function and let* λ *be a nonnegative number. Then* $F(x) = \lambda f(x)$ *is also a concave function. Let* f_1*and* f_2 *be concave functions. Then* $F(x) = f_1(x) + f_2(x)$ *is also concave.*

4 Extrema of concave functions

The importance of concave (and convex) functions from the optimization point of view lies in some properties of concave functions with regard to their extrema.

Theorem 17 ***Local-global property of the maximum***
Let f be a concave function defined on a convex set $C \subset \mathbb{R}^n$. Then every local maximum of f at $x^* \in C$ is a global maximum of f over all C.

Theorem 18 *The set of points at which a concave function f attains its maximum over C is a convex set.*

Corollary 19 *Let f be a strictly concave function, defined on the convex set $C \subset \mathbb{R}^n$. If f attains its maximum at $x^* \in C$, this maximizing point is unique.*

Theorem 20 ***Sufficiency condition for global extrema***
Let f be a differentiable concave (strictly concave) function on the convex set C. If

$$\nabla f(x^*) = 0$$

at a point $x^* \in C$, then f attains its maximum (unique maximum at x^*).

5 Quasi-concave and quasi-convex functions

We begin generalizing concave functions by recalling from the preceding chapter that the upper-level sets of concave functions are convex sets. Concavity of a function is a sufficient condition for this property, but not a necessary one. We define a family of functions by the convexity of their upper-level sets. Such functions are called quasi-concave functions. They are generalized concave functions, since it is easy to show that every concave function is quasiconcave, but not conversely.

5.1 Quasi-concave functions

Definition 21 *Let f be defined on the convex set $C \subset \mathbb{R}^n$. It is said to be **quasiconcave** if its upper-level sets*

$$U(f, \alpha) = \{x : x \in C, f(x) \geq \alpha\}$$

*are convex sets for every real α. Similarly, f is said to be **quasiconvex** if its lower-level set*

$$L(f, \alpha) = \{x : x \in C, f(x) \leq \alpha\}$$

are convex sets for every real α.

It is a straightforward exercise to show the following alternative characterization for quasiconcave functions.

Theorem 22 *Let f be defined on the convex set $C \subset \mathbb{R}^n$. It is a quasiconcave function if and only if*

$$f(tx_1 + (1-t)x_2) \geq \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in C$,and $0 \leq t \leq 1$.

Note that the preceding characterization of quasiconcavity is identical to the following one

$$f(x_1) \geq f(x_2) \Rightarrow f(tx_1 + (1-t)x_2) \geq f(x_2).$$

From this, it follows immediately that all monotonic functions of a real variable are quasiconcave. **Caution:** This does not hold for $f : \mathbb{R}^n \to \mathbb{R}$.

It is also clear that a concave function is also quasiconcave, since

$$f(tx_1 + (1-t)x_2) \geq f(x_1) + (1-t)f(x_2) \geq \min\left[f(x_1), f(x_2)\right].$$

The most important property of quasiconcave functions for microeconomic theory is, however, the following:

Theorem 23 *If $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave and if $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $g(f(x))$ is also quasiconcave.*

The theorem above is in contrast to the behavior of concave functions. Its importance for microeconomics stems from the fact that in consumer theory, preferences of a consumer identify the level sets of any utility function representing the preferences, but not the numerical values of the utility levels. Convexity of preference is equivalent to quasiconcavity of the utility representation and hence the theorem above states that any increasing function of a given utility function is a representation of the same preferences.

Theorem 24 ***(Local-global property of the maximum)***
Let f be a strictly quasiconcave function defined on the convex set $C \subset \mathbb{R}^n$. If $x^* \in C$ is a local maximum of f, then x^* is also a strict global maximum of f on C. The set of points at which f attains its global maximum over C is a convex set.

5.2 Differentiable quasi-concave functions

Theorem 25 ***(Quasiconcavity for C^1 functions)***
Let f be differentiable on the open convex set $C \subset \mathbb{R}^n$. Then f is quasiconcave if and only if for every $x_1, x_2 \in C$

$$f(x_1) \geq f(x_2) \Rightarrow \nabla f(x_2) \cdot (x_1 - x_2) \geq 0.$$

Theorem 26 *Let f be a twice differentiable quasiconcave function on the open convex set $C \subset \mathbb{R}^n$. If $x_0 \in C, v \in \mathbb{R}^n$,*

$$v^T \nabla f(x_0) = 0 \Rightarrow v^T \nabla^2 f(x_0) v \leq 0.$$

The following strengthening of the theorem allows a complete characterization of quasiconcave function for case that $\nabla f(x) \neq 0$ for every $x \in C$.

Theorem 27 *Let f be a twice differentiable quasiconcave function on the open convex set $c \subset \mathbb{R}^n$ and suppose that $\nabla f(x) \neq 0$ for every $x \in C$. Then f is quasiconcave if and only if $x \in C, v \in \mathbb{R}^n$,*

$$v^T \nabla f(x_0) = 0 \Rightarrow v^T \nabla^2 f(x_0) v \leq 0.$$

We conclude by mentioning some more necessary and sufficient conditions for quasiconcavity of twice differentiable functions - this time in terms of "bordered determinants" or "bordered Hessians."

Theorem 28 ***(Quasiconcavity for C^2 functions)***

1. *Let f be a twice differentiable function on the open convex set $C \subset \mathbb{R}^n_+$ (the non-negative orthant). Define the determinants $\mathcal{D}_k(x), k+1, ..., n$ by*

$$\mathcal{D}_k(x) = \begin{vmatrix} 0 & \frac{\partial f}{\partial x_i} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{x_I} & \frac{\partial^2 f}{\partial s_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \cdots & & & \cdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_k} \end{vmatrix}.$$

   A necessary condition for f to be quasi-concave is that $(-1)^k \mathcal{D}_k(x) \geq 0$ for all $k = 1, ..., n$ and all $x \in C$.

2. *A sufficient condition for f to be quasi-concave is that $(-1)^k \mathcal{D}_k(x) > 0$ for all $k = 1, ..., n$ and all $x \in C$.*

5.3 Strict Quasiconcavity

Definition 29 *Let f be defined on the convex set $C \subset \mathbb{R}^n$. It is said to be strictly quasiconcave if*

$$f(tx_1 + (1-t)x_2) > \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in C, x_1 \neq x_2$, *and* $0 < t < 1$. *If* f *is strictly quasiconcave, then* $g \equiv -f$ *is a strictly quasiconvex function.*[1]

What is the difference between quasiconcave and strictly quasiconcave functions. A function that is quasiconcave, but not strictly quasiconcave is constant on some interval of its domain of definition. Note also that strict quasiconcavity is not a proper generalization of concavity, but only of strict concavity.

Theorem 30 *Let* f *be a continuous function, defined on* $\mathbb{R}^n$. *If* f *is strictly quasiconcave, then its upper level sets are strictly convex.*

[1] Currently there are several competing definitions of strict quasiconcavity. This one is the one most commonly used in economics.