FDPE Math Camp

Exercise Set 4

Suggested Solutions

1. Concave/quasiconcave functions and convex sets

   (a) Consider the CES-function

   $$f(x_1, x_2) = A\left(\delta x_1^\rho + (1-\delta)\, x_2^\rho\right)^{1/\rho}$$

   First note that the functions x_1^ρ and x_2^ρ are both concave. Since $0 < \delta < 1$, also $\delta x_1^\rho + (1-\delta)\, x_2^\rho$ is by theorem 21.8 concave (note: $\rho \in [0,1]$). Now since $(\cdot)^{1/\rho}$ and multiplication by a positive constant A are monotone transformations, $f(x_1, x_2)$ is a monotone transformation of a concave function as is therefore quasiconcave.

   (b) We will prove by contradiction that if a quasiconcave function has a maximum, it is unique. To this end, suppose that the function obtains its maximum value at both x and y where $x \neq y$. Clearly we must then have that $f(x) = f(y)$. Then quasiconcavity implies that for any $\alpha \in (0,1)$

   $$f(\alpha x + (1-\alpha)\, y) > \min\{f(x), f(y)\}.$$

   This provides, however, a contradiction to the fact that x and y are maxima. Hence, the maximum must be unique.

   (c) The convexity of $f^{-1}(S)$ can be shown by using the definition of convexity. Take two points x^1 and x^2 from $f^{-1}(S)$ and form their convex combination $x^\lambda = \lambda x^1 + (1-\lambda)x^2$ for $\lambda \in [0,1]$. By the definition of f it holds that $y^i = Ax^i + b$ for some $y^i \in S$, $i = 1, 2$. For $y^\lambda = \lambda y^1 + (1-\lambda)y^2$ we then have $y^\lambda \in S$ by the convexity of S. On the other hand, $y^\lambda = \lambda(Ax^1 + b) + (1-\lambda)(Ax^2 + b) = A(\lambda x^1) + A((1-\lambda)x^2) + (\lambda + 1 - \lambda)b = A(\lambda x^1 + (1-\lambda)x^2) + b$, which means that y^λ is the image of x^λ. Hence, $x^\lambda \in f^{-1}(S)$, which completes the proof.

2. We are asked to find the optimum for the constraint maximization problem

   $$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x \quad s.t. \quad Ax = b$$

   To this end, we will write the Lagrangian

   $$L(x, \lambda) = \frac{1}{2} x^T Q x - \lambda^T (Ax - b)$$

   First-order conditions

   $$\nabla L(x, \lambda) = Qx - A^T \lambda = 0 \iff x = Q^{-1} A^T \lambda$$

   Furthermore, substituting in the constraint yields

   $$AQ^{-1}A^T \lambda = b \iff \lambda = \left(AQ^{-1}A^T\right)^{-1} b$$

   Note that since Q is symmetric and positive definite, so is Q^{-1}, and because A has full rank the matrix $AQ^{-1}A^T$ is invertible (as a positive definite matrix). Now we can substitute in FOC

   $$x = Q^{-1}A^T \left(AQ^{-1}A^T\right)^{-1} b$$

   Moreover, second-order conditions are given by

   $$\nabla^2 L(x, \lambda) = Q$$

   since Q is positive definite, we indeed have a minimum.

3. Consider the optimization problem

$$\max_{x,y} \alpha x + \sqrt{y}$$

$$\begin{aligned} \text{s.t.} \quad px + y &\leq 1, \quad p > 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

i) Lagrangian for the problem

$$L(x, y, \lambda) = \alpha x + \sqrt{y} + \lambda_1 (1 - px - y) + \lambda_2 x + \lambda_3 y$$

First-order conditions

$$\begin{aligned} \frac{\partial L(x,y,\lambda)}{\partial x} &= \alpha - \lambda_1 p + \lambda_2 = 0 \\ \frac{\partial L(x,y,\lambda)}{\partial y} &= \frac{1}{2\sqrt{y}} - \lambda_1 + \lambda_3 = 0 \\ px + y &\leq 1 \\ \lambda_1 (1 - px - y) &= 0 \\ x &\geq 0 \\ y &\geq 0 \\ \lambda_2 x &= 0 \\ \lambda_3 y &= 0 \end{aligned}$$

ii) Suppose that the first constraint will bind and the other two will be slack at the optimum. We will then have

$$\begin{aligned} \alpha &= \lambda_1 p \\ \frac{1}{2\sqrt{y}} &= \lambda_1 \\ px + y &= 1 \end{aligned}$$

By dividing

$$\begin{aligned} \sqrt{y} &= \frac{p}{2\alpha} \implies y = \left(\frac{p}{2\alpha}\right)^2 \\ x &= \frac{1-y}{p} = \frac{1 - \left(\frac{p}{2\alpha}\right)^2}{p} = \frac{4\alpha^2 - p^2}{4\alpha^2 p} \end{aligned}$$

If $4\alpha^2 - p^2 \leq 0$, i.e., α is small relative to p, then the optimal solution is $x = 0$, $y = 1$. On the other hand, if $4\alpha^2 - p^2 \geq 1$, then the optimum is obtained at $x = 1$, $y = 0$ (note that $\sqrt{y}$ is not differentiable at $y = 0$ so the first order conditions are not defined at this corner point).

iii) The solution is an optimum because the objective function is concave, feasible set is convex, and the non-degeneracy constraint qualification holds.

You may also check the definiteness of the bordered Hessian for the problem

$$H = \begin{pmatrix} 0 & \frac{\partial h(x,y,\lambda)}{\partial x} & \frac{\partial h(x,y,\lambda)}{\partial y} \\ \frac{\partial h(x,y,\lambda)}{\partial x} & \frac{\partial^2 L(x,y,\lambda)}{\partial x^2} & \frac{\partial^2 L(x,y,\lambda)}{\partial xy} \\ \frac{\partial h(x,y,\lambda)}{\partial y} & \frac{\partial^2 L(x,y,\lambda)}{\partial yx} & \frac{\partial^2 L(x,y,\lambda)}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 & p & 1 \\ p & 0 & 0 \\ 1 & 0 & -\frac{1}{4y\sqrt{y}} \end{pmatrix}$$

Moreover, $\det H = p^2 y^{-2/3} > 0$ such that we have a maximum.

4. Production maximization problem

   (a) The firm chooses an optimal combination of capital and labor in order to

$$\max_{K,L} Q(K,L) = 50K^2\sqrt{L}$$

$$\text{s.t.} \quad K + L = 80$$

   Lagrangian

$$L(K,L,\lambda) = 50K^2\sqrt{L} + \lambda(80 - K - L)$$

   First-order conditions

$$\begin{aligned}
\frac{\partial L(K,L,\lambda)}{\partial K} &= 100K\sqrt{L} - \lambda = 0 \\
\frac{\partial L(K,L,\lambda)}{\partial L} &= 25\frac{K^2}{\sqrt{L}} - \lambda = 0 \\
\frac{\partial L(K,L,\lambda)}{\partial \lambda} &= 80 - K - L = 0
\end{aligned}$$

   Dividing the first two yields

$$\begin{aligned}
K &= 4L \\
L &= 80 - K
\end{aligned}$$

   That is,

$$\begin{aligned}
K &= 64 \\
L &= 16
\end{aligned}$$

   The optimal output is then

$$Q^{\max} = 50 \cdot 64^2 \cdot \sqrt{16} \approx 819,200$$

   (b) Langrangian multiplier can be interpreted as marginal product of money.[1] From the first-order conditions, we can solve

$$\lambda = 100K\sqrt{L} = 100 \cdot 64 \cdot \sqrt{16} \approx 25600$$

   Thus, if we decrease the budget by 1000 euros, the approximated change of the output is

$$\Delta Q \approx \lambda.$$

   (c) The optimal inputs under the new budget constraint are

$$\begin{aligned}
K &= 63.2 \\
L &= 15.8
\end{aligned}$$

   The real difference in output is

$$Q^{\max} - Q^{\text{new}} = 50 \cdot 64^2 \cdot \sqrt{16} - 50 \cdot 63.2^2 \cdot \sqrt{15.8} \approx 25361$$

5. Utility maximization problem

   Only the budget constraint will bind at optimum, such that the consumer will

$$\max_{c_0,c_1} [u(c_0) + \delta u(c_1)]$$

$$\text{s.t.} \quad c_1 = (1+r)(w_0 - c_0)$$

[1] See e.g. Ch 19 SB for details.

We will first derive the prices for consumption in each period. The idea here is to see that the gross interest rate is actually the ratio of the price for consumption today and consumption tomorrow. That is,

$$1 + r = \frac{p_0}{p_1}$$

Thus, we can rewrite the budget constraint as

$$c_1 = \frac{p_0}{p_1}(w_0 - c_0) \iff p_0c_0 + p_1c_1 = p_0w$$

(a) Using the budget constraint we currently derived, we can write the Lagrangian as

$$L(c_0, c_1, \lambda) = u(c_0) + \delta u(c_1) + \lambda[p_0w_0 - p_0c_0 - p_1c_1]$$

First-order conditions

$$\begin{aligned} u'(c_0) - \lambda p_0 &= 0 \\ \delta u'(c_1) - \lambda p_1 &= 0 \end{aligned}$$

Combining yields

$$\frac{p_0}{p_1} = \frac{u'(c_0)}{\delta u'(c_1)} = MRS(c_0, c_1)$$

(b) The solution is globally optimal if the problem is a concave optimization problem. This is the case, when $u(\cdot)$ is concave. Note that the budget set is convex.
For local maximum, the sufficient condition is that $u''(c_0^*, c_1^*) < 0$. The bordered Hessian for the problem will then have a positive determinant (check!).

6. Expenditure minimization problem

$$\min_x p \cdot x \quad \text{s.t.} \quad U(x) \geq u, x \geq 0$$

(a) By quasiconcavity of $U(\cdot)$, we have that the set of feasible solutions is here convex. This follows from the fact that U is quasiconcave if and only if its upper level sets are convex. The set of feasible consumption allocations is simply the one of these upper level sets. Furthermore, the objective function here is linear, so it is concave. Thus, the expenditure minimization problem is a convex optimization problem.

(b) Recall that Weierstrass theorem implies that a continuous function obtains its minimal value over a compact set. Now the set of feasible solutions is closed but not bounded. However, note that we are looking for a minimum. Thus, we will choose some arbitrary consumption allocation x^* that satisfies $U(x^*) \geq u$ and $p \cdot x \leq p \cdot x^*$. Weierstrass theorem will then hold for any $p >> 0$.

(c) For the Cobb-Douglas utility function, the expenditure minimization problem will take the form

$$\min_{x_1, x_2} p_1x_1 + p_2x_2 \quad \text{s.t.} \quad x_1^\alpha x_2^{1-\alpha} \geq u$$

Lagrangian for the problem

$$L(x_1) = p_1x_1 + p_2x_2 + \lambda\left(u - x_1^\alpha x_2^{1-\alpha}\right)$$

The FOC

$$p_1 - \lambda\alpha x_1^{\alpha-1}x_2^{1-\alpha} = 0$$
$$p_2 - \lambda(1-\alpha)x_1^\alpha x_2^{-\alpha} = 0$$

Combining yields (solve for λ from either of the equations and plug to the other one)

$$x_1 = \frac{\alpha}{1-\alpha}\frac{p_2}{p_1}x_2$$

Substituting in the constraint

$$\left(\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}x_2\right)^{\alpha} x_2^{1-\alpha} = u$$

$$\iff x_2 = \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} u$$

Similarly,

$$x_1 = \left(\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\right)^{1-\alpha} u$$

Note that $x_1(p_1, p_2, u)$ and $x_2(p_1, p_2, u)$ are the celebrated Hicksian demand functions for the two goods.